EXTENDED SELF SIMILARITY IN NUMERICAL SIMULATIONS OF 3D ANISOTROPIC TURBULENCE.

R. BENZI[1], M. V. STRUGLIA[1], R. TRIPICCIONE[2]

[1] Dipartimento di Fisica,
Università di Roma *Tor Vergata*,
Via della Ricerca Scientifica 1, 00133 Roma, Italy
and
Infn, Sezione di Roma *Tor Vergata*
[2] Infn, Sezione di Pisa, S. Piero a Grado, 50100 Pisa, Italy

Abstract

Using a code based on the Lattice Boltzmann Equation, we have performed numerical simulations of a turbulent shear flow. We investigate the scaling behaviour of the structure functions in presence of *anisotropic* homogeneous turbulence, and we show that although Extended Self Similarity does not hold when strong shear effects are present, a more generalized scaling law can still be defined.

In the last few years there has been a growing attention on the scaling properties of *fully developed turbulence* and, in particular, on the characterization of the Probability Distribution Function of the velocity increments $\delta_r v \equiv v_x(\underline{x}+\underline{r}) - v_x(\underline{x})$, i.e. the velocity difference in the $x-$direction between two points at distance r.

To this aim, usually one considers the scaling properties of the structure functions defined as:

$$F_n(r) = \langle |\delta_r v|^n \rangle. \tag{1}$$

According to the Kolmogorov theory[1] a scaling law for (1) is expected to hold in the so called inertial range, $\eta \ll r \ll L$,(L being the integral scale of the flow and η the Kolmogorov scale):

$$F_n(r) = A_n \left(\epsilon r \right)^{\frac{n}{3}} \tag{2}$$

where A_n are dimensionless constants and ϵ is the mean rate of energy dissipation.

There has been many experimental and numerical results suggesting that, because of the *intermittency* of the velocity field, the relation (2) is violated, giving an *anomalous scaling law* with scaling exponents $\zeta_n \neq \frac{n}{3}$.

By taking into account the fluctuations of the energy dissipation field, the equation (2) has been modified by Kolmogorov[2] who introduced the *Refined Similarity Hypothesis* (RSH):

$$F_n(r) = A_n \langle \epsilon_r^{\frac{n}{3}} \rangle r^{\frac{n}{3}} \tag{3}$$

where ϵ_r is the local rate of energy transfer.

$$\epsilon_r \equiv \frac{1}{r^3} \int_{B(r)} \epsilon(\underline{x}) d^3 x.$$

At present, most of the efforts, both theoretical and experimental, are devoted to the determination of the anomalous scaling exponents and to the investigation of the role played by the RSH.

The aim of this work is to investigate the scaling properties of the structure functions in the case of a homogeneous shear flow, as a simple example of anisotropic homogeneous turbulence. We are mainly interested to study the scaling laws of the structure functions and to establish if the Extended Self Similarity, recently introduced in literature[3, 5], still holds for shear flows,

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i.e. in presence of a non isotropic turbulent flow.

In this Letter we first remind some concepts about Extended Self Similarity (E.S.S.) and its relevance in order to estimate the ζ_n.

Next we briefly describe the shear flows and some of their properties. Finally we discuss the numerical simulation and show that E.S.S does not hold for shear flows, while a generalized scaling law, involving both E.S.S. and R.S.H. is valid.

In principle, we can determine the scaling exponents ζ_n by means of experimental and numerical measures, but in the latter case some technical problems arise.

The highest Reynolds numbers that can be achieved by laboratory experiments are about $10^{6 \div 7}$, while the numerical simulations performed with the most powerful computers now avalaible can reach $Re \sim 10^3$. As the computational effort grows like Re^3, it could seem very hard to obtain good estimates, at least comparable to the experimental results, of the scaling exponents by the numerical simulations.

The concept of Extended Self Similarity (E.S.S.) can help us to fill up this gap.

The idea is to investigate the scaling behaviour of one structure function against the other, namely

$$F_n(r) \sim F_m(r)^{\beta(n,m)} \tag{4}$$

In particular it is expected that, at least in the inertial range, $\beta(n, 3) = \zeta_n$.

Actually, there is strong evidence that E.S.S. is a powerful tool to investigate the scaling laws and that it has many advantages respect to the usual scaling against r, namely:

- it holds down to the dissipative range $r \sim 4 \div 5\eta$,
- it holds also for low Reynolds numbers.

Last but not least, the two previous properties allow a very accurate determination of the scaling exponents. Indeed, the ζ_n can be estimated with an error of just a few percent.

The above statements can be summarized as follows. We can always write the structure functions in the following way:

$$F_p(r) = C_p U_0^p \left[\frac{r}{L} f_p \left(\frac{r}{\eta} \right) \right]^{\zeta_p} \tag{5}$$

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with $U_0^3 = F_3(r)$, $L = U_0^3/\epsilon$ being the integral scale, and C_p dimensionless constants selected in such a way that $f_p(r/\eta) = 1$ for $r \gg \eta$.

E.S.S. implies that, for all the orders p, the function $f_p(r/\eta) \equiv f(r/\eta)$ is the same.

We want to understand which are the effects of the lack of isotropy on the anomalous scaling law defined in (4). To this effect, we consider a simple shear flow.

Let us consider the usual Navier-Stokes equations describing a viscous, incompressible fluid of density ρ, and velocity field $\vec{v}(\underline{x}, t)$:

$$\partial_t \vec{v} + (\vec{v} \cdot \vec{\nabla})\vec{v} = -\frac{1}{\rho}\vec{\nabla}p + \nu\Delta\vec{v} + \vec{f} \tag{6}$$

$$\vec{\nabla} \cdot \vec{v} = 0$$

Let us indicate the stationary solution of the above equations as $\vec{U}$, and define the turbulent velocities $\vec{w}$ as:

$$\vec{v} = \vec{U} + \vec{w}. \tag{7}$$

In order to simplify the following discussion we choose the $x-$direction as the direction of the main flow: $U_x = U$, $U_y = 0$, $U_z = 0$.

We have a *homogeneous shear flow* [6] when the main motion has a constant velocity in a given direction and a constant lateral velocity gradient throughout the whole field, e.g. $U_x = U(z)$ and $\frac{dU_x}{dz} = S$, so there is an evident lack of isotropy in the system.

Moreover we have a non zero turbulence shear stresses tensor, the component $\langle w_x w_z \rangle$ is different from zero and it makes a positive contribution only to $\partial_t \langle w_x^2 \rangle$, resulting in non isotropy.

A generalization of the "$\frac{4}{5}$" Kolmogorov equation for anisotropic homogeneous shear flow [7] suggest that the typical scale fixed by the shear intensity is $r_s \sim (\epsilon/S^3)^{\frac{1}{2}}$. With zero shear this scale is infinite, otherwise it has a finite value: below this scale the shear effects are expected to become negligible.

The particular question we want to address is: what does it happen to the scaling laws (4) when r_s falls into the inertial range?

In order to answer this question we perform a direct numerical simulation of a turbulent shear flow, using a code based on the *Lattice Boltzmann Equation*, for computational details see, for instance, [8, 9].

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We simulate a 3D fluid occupying a volume of $V = L^3$ sites with $L = 160$, viscosity $\nu = 0.014$, and obeying to the usual N-S equations plus a forcing term $\vec{f} = (f_x(z), 0, 0)$ chosen such that the stationary solution of the N-S equations is:

$$U_x = A sin(k_z z) \qquad U_y = 0 \qquad U_z = 0. \tag{8}$$

$k_z = \frac{8\pi}{L}$ being the wave vector corresponding to the integral scales, and $A = 0.3$.

In this way the shear has a spatial dependence $S(z) \sim cos(k_z z)$. We can access both zones where the shear is maximum and locally homogeneous, and zones where the shear is minimum.

We evaluated v_{rms} as the mean value of $(\frac{2}{3}E)^{1/2}$. The simulations have been done at $Re_\lambda = \frac{\lambda v_{rms}}{\nu} \sim 40$, with $\lambda \sim 15$ lattice spacings, and the Kolmogorov scale is about 1 lattice spacing wide.

The simulation has advanced 100000 iterations corresponding to about 25 macroscale eddy turnover times $\tau_0 \sim L/v_{rms}$: 40 velocity configurations have been saved every 2500 time steps, in order to ensure the statistical independence of the different configurations.

We have evaluated the structure functions $F_n(r)$ up to the tenth order. The mean values of $|\delta_r v|^n$ have been evaluated through time and spatial average at fixed z-level:

$$\langle O(\underline{r}, t) \rangle = \frac{1}{T} \int_0^T dt \; \frac{1}{L^2} \int dx dy \; O(\underline{r}, t).$$

In Fig.1 we have a log-log plot of the longitudinal (x-direction) structure function $F_6(r)$ against $F_3(r)$, obtained from the velocity fields corresponding to the minimum shear level. The dashed curve is the best fit done in the range between the 20-th and 30-th grid point, and corresponds to a slope of 1.79 in good agreement with other measured values of ζ_6. Every point in the plot corresponds to a grid point and the lattice spacing is $\sim 1\eta$ wide. As we can see the E.S.S. holds as usual until $4 \div 5\eta$.

Fig.2 shows the same plot but at the maximum shear level. It is quite evident that E.S.S. does not hold. In any case, the slope corresponding to the best fit can be estimated at about 1.43, quite different from the previous value. Similar results have been obtained for all the others structure functions.

In Table 1 we show the scaling exponents obtained for the even order structure functions.

	ζ_2	ζ_4	ζ_6	ζ_8	ζ_{10}
min sh	0.70	1.28	1.79	2.25	2.68
max sh	0.76	1.18	1.43	1.56	1.61
SL mod.	0.696	1.279	1.778	2.211	2.593

Table 1: Scaling exponents evaluated at the minimum shear (first line), at the maximum shear (second line), and from the She-Leveque[11] model.

We can suggest the following explanation for the different scaling behaviour in presence of shear. In our simulations the scale r_s is about 4 lattice spacings at the maximum shear level, so the entire range over which the E.S.S. holds (see Fig.1) is subjected to the shear effects.

Our result clearly shows that the shear completely destroys the E.S.S.

We now turn our attention to RSH. Following [5] we can consider the generalization of RSH by introducing an *effective scale* $S(r) \equiv \langle \delta_r v^3 \rangle / \langle \epsilon_r \rangle = r f(r/\eta)$. Then ESS combined with RSH suggests:

$$\frac{\delta_r v^3}{S(r)} \sim \epsilon_r \tag{9}$$

If the equation (9) is true, as it has already been verified for experimental data sets referring to homogeneous and isotropic turbulence[5], we expect that the local rate of energy transfer and the structure functions satisfy the following scaling law:

$$\langle \delta_r v^{3n} \rangle \sim \langle \epsilon_r^n \rangle \langle \delta_r v^3 \rangle^n \tag{10}$$

over a range wider than the inertial one.

Using the data from our simulation, we obtained the results shown in Fig.s 3-4.

As we can see the scaling of $\langle \epsilon^2 \rangle \langle |\delta_r v|^3 \rangle^2$ against $\langle \delta_r v^6 \rangle$ is well verified in both the zones of maximum and minimum shear with a slope very close to one. This result is extremely interesting and suggests that the scaling law (10) is universal, regardless the isotropy conditions of the turbulent flow.

Let us summarize the results that have been obtained and suggest a possible interpretation for them and what should be their future developments.

First of all it has been shown that E.S.S. does not hold for anisotropic turbulent flows, according to similar results obtained from experimental data sets of turbulent boundary layers [12], where strong shear effects are expected to appear.

It means that moments of different order show a different dependence from the cutoff scale. This means that the shear affects the function $f_p(r/\eta)$, defined in (5), which is no longer the same for all the orders p.

Nevertheless, the scaling law (10) is valid even in presence of shear and at the smallest scales investigated, suggesting that the scaling law of a generic structure function is related to those of the third one and of the energy dissipation in a universal way, for all analyzed scales, a remarkably non-trivial result.

We think that the investigation of the self-scaling properties of the energy dissipation ϵ_r would deserve more attention, in order to understand how the structure functions of the velocity increments depend on the resolution scale and to explain the ESS violation in shear flows.

A deeper analysis of these arguments, together with other numerical and experimental results, will be the subject for further investigation [13, 14].

References

[1] A. N. Kolmogorov, C. R. Acad. Sci. (USSR) 30 (1941) 299

[2] A. N. Kolmogorov, J. Fluid Mech. 13, 82 (1962)

[3] R.Benzi, S.Ciliberto, R.Tripiccione, C.Baudet, F.Massaioli, S.Succi, *Phys.Rev E* **48** (1993) R29.

[4] M.Briscolini, P.Santangelo, S.Succi, R.Benzi *Phys. Rev. E* **50** (1994) R1745.

[5] R.Benzi, S.Ciliberto, C.Baudet, G.Ruiz Chavarria *Physica D* **80** (1995) 385.

[6] M.M. Rogers, P. Moin *J. Fluid Mech.* **176** (1987), 33-66.

[7] Hinze, "Turbulence. An introduction to its Mechanisms and Theory" Mc Graw-Hill, New York 1959.

[8] U. Frisch, D. d'Humieres, B. Hasslacher, P. Lallemand, Y. Pomeau, J.P. Rivet *Complex Systems* **1** (1987) 649-707.

[9] R. Benzi, S. Succi, M. Vergassola *Physics Reports* 222, 145 (1992)

[10] C. Battista et al. reprinted by G. Parisi in 'Field Theory, Disorder and Simulations', World Scientific (1992), and references therein.

[11] Z. She, E. Leveque Phys.Rev. Lett. **72** 3 (1994).

[12] G. Stolovitzky, K.R. Sreenivasan, Phys.Rev. E **48** 32 (1993).

[13] R. Benzi, L. Biferale, S. Ciliberto, M.V. Struglia, R. Tripiccione *in preparation*

[14] R. Benzi, L. Biferale, S. Ciliberto, M.V. Struglia, R. Tripiccione *in preparation*

Figure Captions.

Figure 1. Log-log plot of $F_6(r)$ against $F_3(r)$ at the minimum shear. The dashed line is the best fit with slope 1.79. Every point in the plot corresponds to a grid point and the lattice spacing is $\sim 1\eta$ wide.

Figure 2. The same as in Fig.1 at the maximum shear. The dashed line is the best fit with slope 1.43.

Figure 3. Plot of $\langle\epsilon^2\rangle\langle|\delta_r v|^3\rangle^2$ against $\langle\delta_r v^6\rangle$ at the minimum shear. The points refer to the scales at $2, 4, 5, 8, 10, 16, 20, 32, 40$ grid points and the dashed line is the best fit done over these points, corresponding to the slope 0.99.

Figure 4. The same as in Fig.3 at the maximum shear. The dashed line is the best fit with slope 0.99.